Lecture Notes, November 6, 2012

General Equilibrium in an Economy with unbounded technology sets

Delete P.VI (bounded $\mathcal{Y}^j$). Like all good mathematicians, we're reducing this to the previous case.

Under assumptions of No Free Lunch (P.IV(a)) and Irreversibility (P.IV(b)), the attainable output set for the economy and for each firm is still bounded.

P.IV. (a) if $y \in Y$ and $y \neq 0$, then $y_k < 0$ for some k.
(b) if .$y \in Y$ and $y \neq 0$, then $-y \notin Y$

Let firm j's (unbounded) production technology be Y^j. Define $S^j(p)$ as j's profit maximizing supply in Y^j. Define $D^i(p)$ as i's demand without restriction to $\{x \mid |x| \leq c\}$ and with income $M^i(p) = p \cdot r^i + \Sigma_j \alpha^{ij}\pi^j(p)$. Note that $S^j(p)$ and $D^i(p)$ may not be well defined.

Define $\tilde{Y}^j = Y^j \cap \{x \mid |x| \leq c\}$, substitute $\tilde{Y}^j$ for $\mathcal{Y}^j$ in chapters 11 - 14. Define $\tilde{S}^j(p)$ as j's supply function based on $\tilde{Y}^j$.

Theorem 15.3(b): If $\tilde{S}^j(p)$ is attainable, then $S^j(p) = \tilde{S}^j(p)$.

Theorem 16.1(b): If $M^i(p) = \tilde{M}^i(p)$, and $\tilde{D}^i(p)$ is attainable, then $\tilde{D}^i(p) = D^i(p)$.

$Z(p) = \Sigma_i D^i(p) - \Sigma_j S^j(p) - \Sigma_i r^i$

Theorem 18.1: Assume P.II-P.V, and C.I-C.V, C.VI(SC), CVII. There is $p^* \in P$ so that p^* is an equilibrium price vector. That is, $Z(p^*) \leq 0$ and $p^*_k = 0$ for k so that $Z_k(p^*) < 0$.

Proof: The artificially bounded economy characterized by production technologies $\tilde{Y}^j$, $j \in F$, is a special case of the bounded economy of chapters 11 - 14. Find equilibrium of that bounded economy. That bounded economy equilibrium is attainable so restriction to length c is not a binding constraint. So bounded and unbounded supply and demand coincide. Equilibrium prices of the bounded economy exist and are equilibrium prices for the unbounded economy with technology sets Y^j. Q.E.D.

Theorem 18.1 here is the most important single result of this course. It says that the competitive economy, guided only by prices, has a market clearing equilibrium outcome. The decentralized price-guided economy has a consistent solution. This is the defining result of the general equilibrium theory.

The Uzawa Equivalence Theorem

Let S be the unit simplex in R^N . Recall two propositions:

Brouwer Fixed Point Theorem (BFPT): Let $f : S \to S$, f continuous. Then there is $p^* \in S$ so that $p^* = f(p^*)$.

Walrasian Existence of Equilibrium Proposition (WEEP):
Let $X : S \to R^N$ so that
(1) X(p) is continuous for all $p \in S$ and
(2) $p \cdot X(p) = 0$ (Walras' Law) for all $p \in S$.[1]
Then there is $p^* \in S$ so that $X(p^*) \leq 0$ with $p_i^* = 0$ for i so that $X_i(p^*) < 0$.

The observation that these two results are equivalent is Theorem 18.2, below. Mathematical equivalence means that each proposition implies the other. We already know that BFPT implies WEEP; that was Theorem 5.2. It remains to demonstrate that the implication goes the other way as well. The proposition requires that ---- using WEEP but not BFPT ---- we prove that for an arbitrary continuous function from the simplex to itself, there is a fixed point. The strategy of proof is to take an arbitrary continuous function f(p) from the simplex into itself. We use f(p) to construct a continuous function mapping from S into R^N fulfilling Walras' Law. That is, we construct an 'excess demand' function (derived from no actual economy but fulfilling the properties required in WEEP). The strategy of proof then is to find the general equilibrium price vector associated with this excess demand function and show that it is also a fixed point for the original function. Obviously this plan requires clever construction of the excess demand function.

[1] We use the strong form of Walras' Law for convenience.

Theorem 18.2 (Uzawa Equivalence Theorem[2]): WEEP implies BFPT.

Proof: We must demonstrate the following property: Let f(•) be an arbitrary continuous function mapping S into S. Assume WEEP but <u>not</u> BFPT. Then there is $p^* \in S$ so that $f(p^*) = p^*$.

Let $f : S \to S$, f continuous.

Let $\mu(p) \equiv \frac{p \cdot f(p)}{|p|^2}$

$\equiv \frac{|p||f(p)|}{|p|^2} \cos(p, f(p)) \leq \frac{|f(p)|}{|p|}$, where cos(p, f(p)) denotes the cosine of the angle included by p, f(p). Let

$$X(p) \equiv f(p) - \mu(p)p \ .$$

X(p) is the 'excess demand' function.

$p \cdot X(p) = p \cdot f(p) - \frac{p \cdot f(p)}{|p|^2} |p|^2 = 0$; this is Walras' Law (2).

Hence, assuming WEEP, there is $p^* \in S$ so that $X(p^*) \leq 0$. Note that by construction $X(p^*) = 0$. This follows since $p_i^* = 0$ for $X_i(p^*) < 0$. If there were i so that $X_i(p^*) < 0$, it would lead to a contradiction: $p_i^* = 0$, so $0 > X_i(p^*) = f_i(p^*) - \mu(p^*)p_i^* = f_i(p^*) - 0 \geq 0$.

Therefore $X(p^*) = f(p^*) - \mu(p^*)p^* = 0$.

So $f(p^*) = \mu(p^*)p^*$. But p^* and $f(p^*)$ are both points of the simplex. The only scalar multiple of a point on the simplex that remains on the simplex occurs when the scalar is unity. That is,
$f(p^*) \in S,\ p^* \in S$ and $f(p^*) = \mu(p^*)p^*$ implies $\mu(p^*) = 1,$ which implies $f(p^*) = p^*$.[3]

Q.E.D.

[2] The result is due to Hirofumi Uzawa (1962).
[3] Acknowledgment and thanks to Jin-lung Lin for providing the central idea of this argument.

The Uzawa Equivalence Theorem says that use of the Brouwer Fixed Point Theorem is not merely one way to prove the existence of equilibrium. In a fundamental sense, it is the only way.